

## EQUIVALENCES AND STRATIFIED FLOPS

SABIN CAUTIS

ABSTRACT. We construct natural equivalences between derived categories of coherent sheaves on the local models for stratified Mukai or Atiyah flops (of type A).

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## 1. INTRODUCTION

**1.1. Overview.** The purpose of this paper is to define in a direct way the kernels which induce equivalences between the (derived) categories of coherent sheaves on the local models for stratified Mukai or Atiyah flops (of type A).

The relation between birational geometry and derived categories of coherent sheaves is an interesting, at times subtle story. On the one hand there are general ideas of when two birational varieties should be  $D$ -equivalent (*i.e.* have isomorphic derived categories). For example, one such conjecture is that  $K$ -equivalence should imply  $D$ -equivalence. Recall that two birational varieties  $Y$  and  $Y'$  are  $K$ -equivalent if they have a common resolution  $Y \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} Y'$  such that  $\pi_1^*(K_Y) \cong \pi_2^*(K_{Y'})$  (see [H] p. 150 for details).

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More generally, given a flip  $X \dashrightarrow Y$  there should be an embedding of  $D(Y)$  into  $D(X)$ . So the minimal model program can be thought of as a process which yields a birational model having the smallest derived category among all models.

On the other hand, one can take specific birational varieties  $Y$  and  $Y'$  and ask for a concrete equivalence between their derived categories. One natural way to construct such an equivalence is to define a birational correspondence

$$\begin{array}{ccc} & Z & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ Y & & Y' \end{array}$$

and hope that the functor  $\pi_{2*} \circ \pi_1^* : D\text{Coh}(Y) \rightarrow D\text{Coh}(Y')$  is an equivalence. Sometimes the obvious correspondence works (*i.e.* it induces an equivalence) and sometimes it does not (see below).

Interesting examples of such correspondences are given by (stratified) Mukai and Atiyah flops (defined below). Such flops appear naturally in higher dimensional birational geometry. For example, Baohua Fu [F] shows that two Springer maps with the same degree over a nilpotent orbit closure are connected by stratified Mukai flops. So if one can show that stratified flops induce equivalences then it follows that any two Springer resolutions are derived equivalent.

Another example is the work with Joel Kamnitzer [CK2] where we use explicit equivalences induced by Mukai flops to construct homological knot invariants. Stratified Mukai flops show up when one tries to generalize this construction in order to categorify the coloured  $sl(m)$  Reshetikhin-Turaev knot invariants [CK3]. In these cases one wants a particular equivalence – it does not suffice to simply know the equivalence exists abstractly.

**1.2. Stratified Mukai and Atiyah flops.** The local model for the **Mukai flop** is a correspondence which relates the cotangent bundles  $T^*\mathbb{P}(V)$  and  $T^*\mathbb{P}(V^\vee)$  of dual projective spaces. Notice that although  $T^*\mathbb{P}(V)$  and  $T^*\mathbb{P}(V^\vee)$  are isomorphic they are not naturally isomorphic (so, for example, they are not isomorphic in families). In particular, this means that the categories of coherent sheaves  $\text{Coh}(T^*\mathbb{P}(V))$  and  $\text{Coh}(T^*\mathbb{P}(V^\vee))$  are not naturally isomorphic.

The correspondence is given by blowing up the zero section in  $T^*\mathbb{P}(V)$  and then blowing down the exceptional divisor in a different direction to obtain  $T^*\mathbb{P}(V^\vee)$ . One can also blow down the zero sections of  $T^*\mathbb{P}(V)$  and  $T^*\mathbb{P}(V^\vee)$  to a common base  $B$ . To summarize, we obtain a commutative diagram

$$\begin{array}{ccc} & W & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ T^*\mathbb{P}(V) & & T^*\mathbb{P}(V^\vee) \\ & \searrow & \swarrow \\ & B. & \end{array}$$

One might guess that  $\pi_{2*} \circ \pi_1^* : D\text{Coh}(T^*\mathbb{P}(V)) \rightarrow D\text{Coh}(T^*\mathbb{P}(V^\vee))$  should give an equivalence. Perhaps a little surprisingly this is not the case [N1]. However, if one uses the fibre product  $Z := T^*\mathbb{P}(V) \times_B T^*\mathbb{P}(V^\vee)$  instead of  $W$  then the induced functor (given by pulling back to  $Z$  and pushing forward) *does* induce a natural equivalence

$$D\text{Coh}(T^*\mathbb{P}(V)) \xrightarrow{\sim} D\text{Coh}(T^*\mathbb{P}(V^\vee))$$

as shown by Kawamata [Ka1] and Namikawa [N1].

Now  $T^*\mathbb{P}(V)$  and  $T^*\mathbb{P}(V^\vee)$  have natural one parameter deformations  $\widetilde{T^*\mathbb{P}(V)}$  and  $\widetilde{T^*\mathbb{P}(V^\vee)}$  over  $\mathbb{A}^1$ . The **Atiyah flop** relates these two spaces by a correspondence also given by blowing up and then

down the zero section. One can also blow down the zero sections to get a common space  $\tilde{B}$  and obtain the analogous commutative diagram

$$\begin{array}{ccc} & \tilde{W} & \\ \tilde{\pi}_1 \swarrow & & \searrow \tilde{\pi}_2 \\ \widetilde{T^*\mathbb{P}(V)} & & \widetilde{T^*\mathbb{P}(V^\vee)} \\ & \searrow & \swarrow \\ & \tilde{B} & \end{array}$$

This time  $W$  is actually the same as the fibre product  $\widetilde{T^*\mathbb{P}(V)} \times_{\tilde{B}} \widetilde{T^*\mathbb{P}(V^\vee)}$  and the functor  $\tilde{\pi}_{2*} \circ \tilde{\pi}_1^*$  induces an equivalence

$$DCoh(\widetilde{T^*\mathbb{P}(V)}) \xrightarrow{\sim} DCoh(\widetilde{T^*\mathbb{P}(V^\vee)}).$$

Mukai flops and Atiyah flops have natural generalizations to cotangent bundles of Grassmannians. More precisely, the **stratified Mukai flop** (of type A) relates  $T^*\mathbb{G}(k, N)$  and  $T^*\mathbb{G}(N - k, N)$  (the standard Mukai flop is the case  $k = 1$ ) via a diagram

$$\begin{array}{ccc} & W & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ T^*\mathbb{G}(k, N) & & T^*\mathbb{G}(N - k, N) \\ & \searrow & \swarrow \\ & B & \end{array}$$

Such flops often show up in the birational geometry of symplectic varieties. We will give a precise description of these spaces and maps in section 2. Perhaps surprisingly, it was shown by Namikawa in [N2] that neither  $W$  nor the fibre product  $Z := T^*\mathbb{G}(k, N) \times_B T^*\mathbb{G}(N - k, N)$  induce an equivalence  $DCoh(T^*\mathbb{G}(2, 4)) \xrightarrow{\sim} DCoh(T^*\mathbb{G}(2, 4))$ . At this point it is not clear what correspondence, if any, one can use to construct this equivalence.

In [CKL3] together with Joel Kamnitzer and Anthony Licata we constructed natural equivalences

$$DCoh(T^*\mathbb{G}(k, N)) \xrightarrow{\sim} DCoh(T^*\mathbb{G}(N - k, N))$$

for all  $0 \leq k \leq N$ . The construction was indirect and involved categorical  $\mathfrak{sl}_2$  actions. However, a little unexpectedly, it yielded a kernel which is a sheaf (*i.e.* a complex supported in one degree). In this paper we identify this sheaf in a more direct way as the pushforward of a line bundle from an open subset of the fibre product  $Z$  (Theorem 3.7).

There is also a deformed version of stratified Mukai flops. These are the **stratified Atiyah flops** (of type A) and they relate  $\widetilde{T^*\mathbb{G}(k, N)}$  and  $\widetilde{T^*\mathbb{G}(N - k, N)}$  (these spaces are natural deformations of  $T^*\mathbb{G}(k, N)$  and  $T^*\mathbb{G}(N - k, N)$  over  $\mathbb{A}^1$ ). It follows from Namikawa's work [N2] that the natural correspondence here (which is the same as the natural fibre product) does not induce an equivalence when  $(k, N) = (2, 4)$ . In the second main theorem of this paper we use the description of the kernel used for the stratified Mukai flop (Theorem 3.7) to construct another kernel (again given via pushforward of a line bundle from an open subset) which induces an equivalence

$$DCoh(\widetilde{T^*\mathbb{G}(k, N)}) \xrightarrow{\sim} DCoh(\widetilde{T^*\mathbb{G}(N - k, N)})$$

for any  $0 \leq k \leq N$  (Theorem 4.1).

In [Ka2] Kawamata wrote down a tweaked version of the functor induced by  $Z$  and showed that it induces an equivalence

$$DCoh(\widetilde{T^*\mathbb{G}(2,4)}) \xrightarrow{\sim} DCoh(\widetilde{T^*\mathbb{G}(2,4)}).$$

We show that our functor agrees with his functor in this special case (Proposition 5.7).

**1.3. Outline.** In section 2 we define all the varieties involved and their relevant deformations. We also review briefly  $S_2$  sheaves and their extensions.

In section 3 we recall the equivalence  $\mathsf{T}(k, N) : DCoh(T^*\mathbb{G}(k, N)) \xrightarrow{\sim} DCoh(T^*\mathbb{G}(N - k, N))$  constructed in [CKL2] and [CKL3] via categorical  $\mathfrak{sl}_2$  actions. We then give a new description of the corresponding kernel  $\mathcal{T}(k, N)$  as a pushforward of a line bundle from a sufficiently large open subset of  $T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(N - k, N)$  (Theorem 3.7).

In section 4 we show that the sheaf  $\mathcal{T}(k, N)$  extends to a sheaf  $\tilde{\mathcal{T}}(k, N)$  on  $\widetilde{T^*\mathbb{G}(k, N)} \times_{\mathbb{A}^1} \widetilde{T^*\mathbb{G}(N - k, N)}$  which induces an equivalence (Theorem 4.1). This sheaf also has a description as the pushforward of a line bundle from an open subset.

Finally, in section 5 we tie up some loose ends. First we discuss a little bit the choices of the line bundles appearing in Theorems 3.7 and 4.1. Then we compute explicitly the inverse of  $\tilde{\mathcal{T}}(k, N)$ . Finally, we explain the relation between our equivalences and the equivalence  $DCoh(\widetilde{T^*\mathbb{G}(2,4)}) \xrightarrow{\sim} DCoh(\widetilde{T^*\mathbb{G}(2,4)})$  constructed by Kawamata in [Ka2].

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## 2. PRELIMINARIES

In this section we discuss notation, define the main varieties we will study and review some facts regarding  $S_2$  sheaves. One can skim this section on a first reading and refer back to it as a reference.

**2.1. Notation.** We will only consider schemes (of finite type) over  $\mathbb{C}$ . If  $Y$  is such a scheme then  $D(Y)$  denotes the bounded derived category of coherent sheaves on  $Y$ . All functors will be derived. So, for example, if  $f : Y \rightarrow X$  is a morphism then  $f_* : D(Y) \rightarrow D(X)$  denotes the derived pushforward (under this convention the plain pushforward is denoted  $R^0f_*$ ).

**Warning 2.1.** There is one important exception to this rule, namely if  $j : U \rightarrow Y$  is an open embedding then  $j_*$  will denote the plain (underived) pushforward.

Recall also briefly the idea of Fourier-Mukai transforms. Given an object  $\mathcal{P} \in D(X \times Y)$  (whose support is proper over  $Y$ ) we may define the associated Fourier-Mukai transform, which is the functor

$$\begin{aligned} \Phi_{\mathcal{P}} : D(X) &\rightarrow D(Y) \\ \mathcal{F} &\mapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P}) \end{aligned}$$

where  $\pi_1$  and  $\pi_2$  are the natural projections from  $X \times Y$ . The object  $\mathcal{P}$  is called the (Fourier-Mukai) kernel.

Fourier-Mukai transforms have right and left adjoints which are themselves Fourier-Mukai transforms. The right adjoint of  $\Phi_{\mathcal{P}}$  is the Fourier-Mukai transform with respect to  $\mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_2^* \omega_X[\dim(X)] \in D(Y \times X)$ . Similarly, the left adjoint of  $\Phi_{\mathcal{P}}$  is the Fourier-Mukai transform with respect to  $\mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_1^* \omega_Y[\dim(Y)]$ , also viewed as a sheaf on  $Y \times X$ . If  $\mathcal{P}$  induces an equivalence  $\Phi_{\mathcal{P}}$  then its inverse is induced by  $\mathcal{P}_L \cong \mathcal{P}_R$ .

**2.2. Varieties.** We will write  $Y(k, N)$  and  $\tilde{Y}(k, N)$  for  $T^*\mathbb{G}(k, N)$  and  $T^*\widetilde{\mathbb{G}(k, N)}$  respectively. If  $Y$  is a variety then  $D(Y)$  will denote the bounded derived category of coherent sheaves on  $Y$ . We usually denote closed immersions by  $i$  and open immersions by  $j$ .

Suppose from now on  $2k \leq N$ . Recall the standard description of cotangent bundles to Grassmannians

$$Y(k, N) = \{(X, V) : X \in \text{End}(\mathbb{C}^N), 0 \xrightarrow{k} V \xrightarrow{N-k} \mathbb{C}^N, X\mathbb{C}^N \subset V \text{ and } XV \subset 0\}$$

where the arrows indicate the codimension of the inclusions. Forgetting  $X$  corresponds to the projection  $Y(k, N) \rightarrow \mathbb{G}(k, N)$  while forgetting  $V$  gives a resolution  $p(k) : Y(k, N) \rightarrow \overline{B(k, N)}$  where  $B(k, N)$  is the nilpotent orbit

$$B(k, N) := \{X \in \text{End}(\mathbb{C}^N) : X^2 = 0 \text{ and } \dim \ker X = N - k\}.$$

The condition  $\dim \ker X = N - k$  is equivalent to  $\text{rank } X = k$  since  $X^2 = 0$ .

Now we also have the resolution  $p(N - k) : Y(N - k, N) \rightarrow \overline{B(k, N)}$ . Thus we get two resolutions

$$Y(k, N) \xrightarrow{p(k)} \overline{B(k, N)} \xleftarrow{p(N-k)} Y(N - k, N)$$

which is the basic example of a **stratified Mukai flop** between  $Y(k, N)$  and  $Y(N - k, N)$ . Consider the fibre product

$$\begin{aligned} Z(k, N) &:= Y(k, N) \times_{\overline{B(k, N)}} Y(N - k, N) \\ &= \{0 \xrightarrow[N-k]{k} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[k]{N-k} \mathbb{C}^N : X\mathbb{C}^N \subset V_1, X\mathbb{C}^N \subset V_2, XV_1 \subset 0, XV_2 \subset 0\}. \end{aligned}$$

The scheme  $Z(k, N)$  consists of  $k + 1$  equi-dimensional components  $Z_s(k, N)$  ( $s = 0, \dots, k$ ) where

$$Z_s(k, N) := \overline{p(k)^{-1}(B(s, N)) \times_{B(s, N)} p(N - k)^{-1}(B(s, N))}.$$

Notice that

$$\{(X, V_1, V_2) \in Z(k, N) : \dim \ker X \geq N - s\} = Z_{k-s}(k, N) \cup Z_{k-s+1}(k, N) \cup \dots \cup Z_k(k, N)$$

$$\{(X, V_1, V_2) \in Z(k, N) : \dim V_1 \cap V_2 \geq s\} = Z_{k-s}(k, N) \cup Z_{k-s-1}(k, N) \cup \dots \cup Z_0(k, N)$$

so we have natural increasing and decreasing filtrations of  $Z(k, N)$ . In particular, one can describe  $Z_s(k, N)$  more directly as

$$Z_s(k, N) = \{(X, V_1, V_2) \in Z(k, N) : \dim \ker X \geq N - k + s \text{ and } \dim V_1 \cap V_2 \geq k - s\}.$$

Since  $\text{span}(V_1, V_2) \subset \ker X$  it follows that  $\dim \ker X + \dim V_1 \cap V_2 \geq N$  on  $Z(k, N)$ . We define the open subscheme

$$Z^o(k, N) := \{(X, V_1, V_2) \in Z(k, N) : N + 1 \geq \dim \ker X + \dim V_1 \cap V_2\} \subset Z(k, N)$$

and  $Z_s^o(k, N) := Z_s(k, N) \cap Z^o(k, N)$ . Abusing notation slightly, we will denote all the open inclusions  $j : Z_s^o(k, N) \rightarrow Z_s(k, N)$ . We will also denote by  $i$  all the closed inclusions of  $Z_s(k, N)$  or  $Z(k, N)$  into  $Y(k, N) \times Y(N - k, N)$ .

On  $Y(k, N)$  we have the natural vector bundle whose fibre over  $(X, V)$  is  $V$ . Abusing notation we denote this vector bundle by  $V$ . Similarly, we have vector bundles  $V_1$  and  $V_2$  on  $Z(k, N)$  as well as quotient bundles  $\mathbb{C}^N/V_1$  and  $\mathbb{C}^N/V_2$ .

We will also use the following natural correspondences between  $Y(k, N)$  and  $Y(k + r, N)$

$$\begin{aligned} W^r(k, N) := \{(X, V_1, V_2) : X \in \text{End}(\mathbb{C}^N), 0 \xrightarrow{k} V_1 \xrightarrow{r} V_2 \xrightarrow{N-k-r} \mathbb{C}^N, \\ X\mathbb{C}^N \subset V_1 \text{ and } XV_2 \subset 0\}. \end{aligned}$$

The two projections give us an embedding  $W^r(k, N) \subset Y(k, N) \times Y(k+r, N)$ . Notice that if  $r = N - 2k$  then we get the standard birational correspondence

$$Y(k, N) \leftarrow W^{N-2k}(k, N) \rightarrow Y(N-k, N)$$

relating stratified Mukai flops.

**2.3. Deformed varieties.** The varieties  $Y(k, N) = T^*\mathbb{G}(k, N)$  have natural one parameter deformations

$\tilde{Y}(k, N) = \{(X, V, x) : x \in \mathbb{C}, X \in \text{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = k \text{ and } \mathbb{C}^N \xrightarrow{X-x \cdot \text{id}} V \xrightarrow{X+x \cdot \text{id}} 0\}$  over  $\mathbb{A}^1$ . The whole picture from the previous section can be repeated for these deformed varieties.

We can look at

$$\tilde{B}(s, N) := \{X \in \text{End}(\mathbb{C}^N) : X^2 = t^2 \cdot \text{id} \text{ and } \dim \ker(X - x \cdot \text{id}) = N - s\}$$

and define  $\tilde{p}(k) : \tilde{Y}(k, N) \rightarrow \overline{\tilde{B}(k, N)}$  as the map which forgets  $V$  (this is a resolution). We also have the map  $\tilde{p}(N-k) : \tilde{Y}(N-k, N) \rightarrow \overline{\tilde{B}(k, N)}$  which forgets  $X$  (and maps  $x$  to  $-x$ ). As before, we get two resolutions

$$\tilde{Y}(k, N) \xrightarrow{\tilde{p}(k)} \overline{\tilde{B}(k, N)} \xleftarrow{\tilde{p}(N-k)} \tilde{Y}(N-k, N)$$

which is the basic example of a **stratified Atiyah flop** between  $\tilde{Y}(k, N)$  and  $\tilde{Y}(N-k, N)$ .

The variety  $Z(k, N)$  also deforms to

$$\begin{aligned} \tilde{Z}(k, N) &:= \tilde{Y}(k, N) \times_{\overline{\tilde{B}(k, N)}} \tilde{Y}(N-k, N) \\ &= \{0 \xrightarrow[N-k]{k} V_1 \xrightarrow[k]{N-k} \mathbb{C}^N : (X - x \cdot \text{id})\mathbb{C}^N \subset V_1, (X + x \cdot \text{id})\mathbb{C}^N \subset V_2 \\ &\quad (X + x \cdot \text{id})V_1 \subset 0, (X - x \cdot \text{id})V_2 \subset 0\}. \end{aligned}$$

However,  $\tilde{Z}(k, N)$  is now irreducible and there are no deformed analogues of  $Z_s(k, N)$ . Notice that  $\tilde{Z}(k, N)$  is naturally a subscheme of  $\tilde{Y}(k, N) \times_{\mathbb{A}^1} \tilde{Y}(N-k, N)$  (where the second projection  $\tilde{Y}(N-k, N) \rightarrow \mathbb{A}^1$  maps  $(X, V, t) \mapsto -t$ ).

Next, we can define the open subscheme

$$\tilde{Z}^o(k, N) := \{(X, V_1, V_2, t) \in \tilde{Z}(k, N) : N+1 \geq \dim \ker(X - x \cdot \text{id}) + \dim V_1 \cap V_2\} \subset \tilde{Z}(k, N).$$

Notice that if  $x \neq 0$  then  $V_1$  and  $V_2$  are uniquely determined by  $X$  as the kernels of  $X + x \cdot \text{id}$  and  $X - x \cdot \text{id}$  respectively. In particular, this means that  $\tilde{Z}^o(k, N)$  contains all the fibres over  $x \neq 0$  since when  $x \neq 0$  we have  $V_1 \cap V_2 = 0$ . We denote the open inclusion  $\tilde{j} : \tilde{Z}^o(k, N) \rightarrow \tilde{Z}(k, N)$ . Also, we denote by  $\tilde{i}$  the closed inclusion of  $\tilde{Z}(k, N)$  into  $\tilde{Y}(k, N) \times_{\mathbb{A}^1} \tilde{Y}(N-k, N)$ . As before we also have natural vector bundles  $V_1, V_2, \mathbb{C}^N/V_1$  and  $\mathbb{C}^N/V_2$ .

Finally, note that the varieties  $W^r(k, N)$  do not deform. The analogue of the standard birational correspondence  $Y(k, N) \leftarrow W^{N-2k}(k, N) \rightarrow Y(N-k, N)$  is just

$$\tilde{Y}(k, N) \leftarrow \tilde{Z}(k, N) \rightarrow \tilde{Y}(k, N)$$

relating stratified Atiyah flops.

**2.4.  $\mathbb{C}^*$  actions.** All varieties in the previous section carry a natural  $\mathbb{C}^*$  action. For  $t \in \mathbb{C}^*$ , this is given by scaling  $X \mapsto t \cdot X \in \text{End}(\mathbb{C}^N)$  and similarly (for the deformed varieties)  $x \mapsto t \cdot x \in \mathbb{A}^1$ .

We will always work  $\mathbb{C}^*$  equivariantly in this paper. In other words  $D(Y)$  will denote the bounded derived category of  $\mathbb{C}^*$ -equivariant coherent sheaves on  $Y$ . Although we could avoid working equivariantly there are a couple of places where the  $\mathbb{C}^*$ -equivariance makes our lives easier since it reduces the dimension of some Hom spaces. We will emphasize the few places where  $\mathbb{C}^*$ -equivariance is actually used.

**2.5. Extensions of  $S_2$  sheaves.** By a scheme we mean a separated scheme of finite type over  $\mathbb{C}$ . We review some known facts about  $S_2$  sheaves.

Let  $Y$  be a scheme and  $\mathcal{F}$  a coherent sheaf on  $Y$ . Recall that if  $Y$  is normal (or  $S_2$ ) then  $\mathcal{F}$  is  $S_2$  if and only if it is torsion free and reflexive. Equivalently,  $\mathcal{F}$  is  $S_2$  if and only if for any open  $U \subset Y$

$$H_X^i(U, \mathcal{F}|_U) = 0$$

where  $X \subset U$  is a locally closed subscheme and  $0 \leq i \leq \min(1, \text{codim}(X) - 1)$ . We will use both of these descriptions.

**Lemma 2.2.** *Let  $U, Y$  be schemes with  $j : U \rightarrow Y$  an open embedding such that  $Z := Y \setminus U \subset Y$  has codimension at least two. If  $\mathcal{G}$  is an  $S_2$  coherent sheaf on  $Y$  then  $\mathcal{G} \cong j_* j^* \mathcal{G}$ . Similarly, if  $\mathcal{F}$  is a coherent sheaf on  $U$  then  $\mathcal{F} \cong j^* j_* \mathcal{F}$ .*

*Proof.* We use the standard long exact sequence of local cohomology

$$0 \rightarrow H_Z^0(Y, \mathcal{G}) \rightarrow H^0(Y, \mathcal{G}) \rightarrow H^0(U, j^* \mathcal{G}) \rightarrow H_Z^1(Y, \mathcal{G}) \rightarrow \dots$$

Since  $\mathcal{G}$  is  $S_2$  and  $\text{codim}(Z) \geq 2$  we have  $H_Z^0(Y, \mathcal{G}) = 0$  and  $H_Z^1(Y, \mathcal{G}) = 0$ . Thus  $H^0(Y, \mathcal{G}) \xrightarrow{\sim} H^0(U, j^* \mathcal{G})$ . But we can replace  $Y$  by any other open set  $V \supset U$  and get the same isomorphism. Thus  $\mathcal{G} = j_* j^* \mathcal{G}$ .

Now consider the composition of adjoint maps

$$j_* \mathcal{F} \rightarrow (j_* j^*) j_* \mathcal{F} = j_* (j^* j_*) \mathcal{F} \rightarrow j_* \mathcal{F}.$$

The left-most map is an isomorphism by the result above while the composition is the identity by the formal properties of adjoint maps. Thus the right-most map is also an isomorphism. Now consider the exact triangle  $j^* j_* \mathcal{F} \xrightarrow{\phi} \mathcal{F} \rightarrow \text{Cone}(\phi)$ . Applying  $j_*$  we get that  $j_* \phi$  is an isomorphism so  $j_* \text{Cone}(\phi) = 0$ . This means  $\text{Cone}(\phi) = 0$  and hence  $\phi$  is an isomorphism.  $\square$

**Proposition 2.3.** *Let  $U, Y$  be schemes with  $j : U \rightarrow Y$  an open embedding such that  $Y \setminus U \subset Y$  has codimension at least two. If  $\mathcal{F}$  is an  $S_2$  coherent sheaf on  $U$  then  $j_* \mathcal{F}$  is a coherent  $S_2$  sheaf on  $Y$ . Moreover, it is the unique such  $S_2$  sheaf whose restriction to  $U$  is  $\mathcal{F}$ .*

*Proof.* Let us assume  $U$  and  $Y$  are both  $S_2$ .

First, since  $j^* \mathcal{O}_Y \cong \mathcal{O}_U$  then by Lemma 2.2 we have  $\mathcal{O}_Y = j_* j^* \mathcal{O}_Y \cong j_* \mathcal{O}_U$ . Thus  $j_* \mathcal{F}$  is a coherent sheaf because  $j_* \mathcal{O}_U \cong \mathcal{O}_Y$ .

Now a sheaf on  $Y$  is  $S_2$  if it is torsion-free and reflexive. Now the (underived) double dual  $(j_* \mathcal{F})^{\vee\vee}$  is reflexive and torsion-free by construction. Also, since  $\mathcal{F}$  is  $S_2$  this means that  $(j_* \mathcal{F})^{\vee\vee}$  is isomorphic to  $j_* \mathcal{F}$  on  $U$ . Thus

$$(j_* \mathcal{F})^{\vee\vee} \cong j_* j^* (j_* \mathcal{F})^{\vee\vee} \cong j_* j^* j_* \mathcal{F} \cong j_* \mathcal{F}$$

where the first and last isomorphisms are by Lemma 2.2. Thus  $j_* \mathcal{F}$  is an  $S_2$  coherent sheaf. To show uniqueness, suppose  $\mathcal{G}$  is another  $S_2$  coherent extension of  $\mathcal{F}$ . Then  $j_* \mathcal{F} \cong j_* j^* \mathcal{G} \cong \mathcal{G}$  where the second isomorphism follows by Lemma 2.2.

Finally, if  $U$  is not  $S_2$  then restrict to an open subscheme  $U' \subset U$  whose complement has codimension at least two and repeat the argument above. If  $Y$  is not  $S_2$  then look at its  $S_2$ -ification  $p : Y' \rightarrow Y$  and apply the argument above to conclude that  $j'_* \mathcal{F}$  is  $S_2$  where  $j' : U' \rightarrow Y'$ . Then  $j_* \mathcal{F} = p_* j'_* \mathcal{F}$  is  $S_2$  since  $p$  is a finite map (c.f. Proposition 5.4 of [KM]).  $\square$

We will also need the following two technical Lemmas.

**Lemma 2.4.** *Let  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  be a short exact sequence of coherent sheaves on some variety  $Y$ . If the scheme theoretic supports  $\text{supp}(\mathcal{F}')$  and  $\text{supp}(\mathcal{F}'')$  are reduced and share no common irreducible components then  $\text{supp}(\mathcal{F})$  is also scheme theoretically reduced.*

*Proof.* □

**Lemma 2.5.** *Consider a short exact sequences of sheaves*

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \rightarrow 0$$

*on an arbitrary scheme  $Y$ . If  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are  $S_2$  then  $\mathcal{F}$  is  $S_2$ .*

*Proof.* This follows immediately from the local cohomology description of  $S_2$  sheaves by considering the long exact sequence

$$0 \rightarrow H_Z^0(\mathcal{F}_1) \rightarrow H_Z^0(\mathcal{F}) \rightarrow H_Z^0(\mathcal{F}_2) \rightarrow H_Z^1(\mathcal{F}_1) \rightarrow H_Z^1(\mathcal{F}) \rightarrow H_Z^1(\mathcal{F}_2) \rightarrow \dots$$

for any closed  $Z \subset X$ . □

### 3. EQUIVALENCES OF STRATIFIED MUKAI FLOPS

In [CKL3] we constructed a natural equivalence  $\mathbb{T} : D(Y(k, N)) \xrightarrow{\sim} D(Y(N - k, N))$  induced by a kernel  $\mathcal{T}$ . We begin by studying the construction of  $\mathcal{T}$ .

**3.1. The varieties  $Z_s(k, N)$ .** We begin with some results on  $Z(k, N)$  and its irreducible components. It is helpful to remark that  $Z(k, N)$  is quite singular. However,  $Z^o(k, N)$  is quite well behaved.

**Lemma 3.1.** *Each  $Z_s(k, N)$  has a natural partial resolution*

$$Z'_s(k, N) := \{0 \xrightarrow{k-s} W_1 \xrightarrow[N-2k+s]{s} \begin{matrix} V_1 & \xrightarrow{N-k} \\ & V_2 \end{matrix} \xrightarrow{k} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XV_1 \subset 0, XV_2 \subset 0\} \xrightarrow{\pi_s} Z_s(k, N)$$

where  $\pi_s$  forgets  $W_1$ .  $Z_s^o(k, N)$  is smooth and the restriction of  $\pi_s$  to the preimage of  $Z_s^o(k, N)$  is an isomorphism. The complements of  $Z_s^o(k, N)$  in  $Z'_s(k, N)$  and  $Z_s(k, N)$  have codimension at least two and four respectively.

*Proof.*  $Z_s(k, N)$  has a natural resolution given by

$$Z''_s(k, N) := \{0 \xrightarrow{k-s} W_1 \xrightarrow[N-2k+s]{s} \begin{matrix} V_1 & \xrightarrow{N-2k+s} \\ & V_2 \end{matrix} \xrightarrow{s} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}.$$

This variety is smooth because forgetting  $V_1$  and  $V_2$  gives us a  $\mathbb{G}(s, N - 2k + 2s) \times \mathbb{G}(N - 2k + s, N - 2k + 2s)$  fibration

$$\pi : Z''_s(k, N) \rightarrow \{0 \xrightarrow{k-s} W_1 \xrightarrow{N-2k+2s} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}.$$

which is actually isomorphic to the conormal bundle of the partial flag variety  $\{0 \xrightarrow{k-s} W_1 \xrightarrow{N-2k+2s} W_2 \xrightarrow{k-s} \mathbb{C}^N\}$  embedded inside the product  $\mathbb{G}(k - s, N) \times \mathbb{G}(N - k + s, N)$ .

Now  $Z_s^o(k, N) \subset Z_s(k, N)$  is defined by the open condition

$$\dim \ker X + \dim V_1 \cap V_2 \leq N + 1.$$

Since  $\dim \ker X \geq N - k + s$  and  $\dim V_1 \cap V_2 \geq k - s$  there are two possibilities: either  $\dim \ker X = N - k + s$  or  $\dim V_1 \cap V_2 = k - s$ . In the first case we can recover  $W_2$  as the kernel of  $X$  and  $W_1$  and the image of  $X$  – this gives a local inverse of  $\pi$ . Similarly, in the second case we can recover  $W_1$  as the intersection of  $V_1$  and  $V_2$  and  $W_2$  as the span of  $V_1$  and  $V_2$ . So  $\pi$  is an isomorphism over  $Z_s^o(k, N)$ . In particular, since  $Z''_s(k, N)$  is smooth this means  $Z_s^o(k, N)$  is smooth.

Finally, the complement  $Z'_s(k, N) \setminus Z_s^o(k, N)$  is covered by three pieces:

- $\dim V_1 \cap V_2 \geq k - s + 2$  (where  $s \geq 2$ ) or
- $\dim V_1 \cap V_2 \geq k - s + 1$  and  $\dim \ker X \geq N - k + s + 1$  (where  $s \geq 1$ ) or
- $\dim \ker X \geq N - k + s + 2$  (where  $s \leq k - 1$ ).



The dimension of the first piece can be computed as the dimension of its resolution

$$\{0 \xrightarrow{k-s} W_1 \xrightarrow{2} W'_1 \xrightarrow[N-2k+s-2]{s-2} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s-2]{N-2k+s-2} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}.$$

Now this is a  $\mathbb{G}(s-2, N-2k+2s-2) \times \mathbb{G}(N-2k+s-2, N-2k+2s-2) \times \mathbb{G}(2, N-2k+2s-2)$  bundle over

$$\{0 \xrightarrow{k-s} W_1 \xrightarrow{N-2k+2s-2} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}$$

which is the conormal bundle of the corresponding partial flag inside  $\mathbb{G}(k-s+2, N) \times \mathbb{G}(N-k+s, N)$ . So its dimension is

$$2(s-2)(N-2k+2s-2) + 2(N-2k+2s-4) + (k-s+2)(N-k+s-2) + (N-k+s)(k-s) = 2k(N-k) - 4.$$

Now the dimension of  $Z_s(k, N)$  is  $2k(N-k)$  so the codimension of the first piece is 4.

The codimensions of the second and third pieces are computed similarly. For the second piece we use the resolution

$$\{0 \xrightarrow{k-s} W_1 \xrightarrow{1} W'_1 \xrightarrow[N-2k+s-1]{s-1} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s-1]{N-2k+s-1} W_2 \xrightarrow{k-s+1} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}$$

which has dimension  $2k(N-k) - 2$  (codimension 2).

For the third piece we use the resolution

$$\{0 \xrightarrow{k-s} W_1 \xrightarrow{s} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s+2]{N-2k+s+2} W_2 \xrightarrow{k-s+2} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}$$

which has dimension  $2k(N-k) - 4$  again.

To show that the codimension of  $Z_s(k, N) \setminus Z_s^o(k, N) \subset Z_s(k, N)$  is at least four the same argument as above works except in the second case. There one needs to use the resolution

$$\{0 \xrightarrow{k-s+1} W_1 \xrightarrow[N-2k+s-1]{s-1} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s-1]{N-2k+s-1} W_2 \xrightarrow{k-s-1} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}$$

which has dimension  $2k(N-k) - 2(N-2k+2s)$ . This is of codimension  $2(N-2k+2s) \geq 4s \geq 4$  since  $N \geq 2k$  and  $s \geq 1$ .  $\square$

**Corollary 3.2.**  $\pi_{s*} \mathcal{O}_{Z'_s(k, N)} \cong j_* \mathcal{O}_{Z_s^o(k, N)}.$

*Proof.* We can factor  $j : Z_s^o(k, N) \rightarrow Z_s(k, N)$  as

$$Z_s^o(k, N) \xrightarrow{j'} Z'_s(k, N) \xrightarrow{\pi_s} Z_s(k, N).$$

We will see in the proof of Proposition 3.6 that  $Z'_s(k, N)$  can be expressed as the intersection of the expected dimension of two smooth varieties (inside a smooth ambient variety). This means  $Z'_s(k, N)$  is a local complete intersection. Also, by Lemma 3.1 the codimension of the complement of  $Z_s^o(k, N)$  inside  $Z'_s(k, N)$  is at least two. By Proposition 2.3 this means that  $j'_* \mathcal{O}_{Z'_s(k, N)} \cong \mathcal{O}_{Z_s^o(k, N)}$ . Thus

$$\pi_{s*} \mathcal{O}_{Z'_s(k, N)} \cong \pi_{s*} j'_* \mathcal{O}_{Z'_s(k, N)} \cong j_* \mathcal{O}_{Z_s^o(k, N)}.$$

$\square$

By Lemma 3.1 we can identify  $Z_s^o(k, N)$  with

$$\{0 \xrightarrow{k-s} W_1 \xrightarrow{s} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s]{N-2k+s} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0, \dim \ker X + \dim V_1 \cap V_2 \leq N+1\}.$$

Keeping this in mind we have the following.

**Lemma 3.3.** *The intersection*

$$\begin{aligned} D_{s,s+1}^o(k, N) &:= Z_s^o(k, N) \cap Z_{s+1}(k, N) \\ &= \{(X, V_1, V_2) \in Z_s^o(k, N) : \dim \ker X = N - k + s + 1 \text{ and } \dim V_1 \cap V_2 = k - s\} \end{aligned}$$

is a divisor in  $Z_s^o(k, N)$  which is the locus where  $X : \mathbb{C}^N/W_2 \rightarrow W_1$  is not an isomorphism. More precisely

$$(1) \quad \mathcal{O}_{Z_s^o(k, N)}(D_{s,s+1}^o(k, N)) \cong \det(\mathbb{C}^N/W_2)^\vee \otimes \det(W_1).$$

Similarly, the intersection

$$\begin{aligned} D_{s,s-1}^o(k, N) &:= Z_s^o(k, N) \cap Z_{s-1}(k, N) \\ &= \{(X, V_1, V_2) \in Z_s^o(k, N) : \dim \ker X = N - k + s \text{ and } \dim V_1 \cap V_2 = k - s + 1\} \end{aligned}$$

is a divisor in  $Z_s^o(k, N)$  which is the locus where the inclusion  $V_1/W_1 \rightarrow W_2/V_2$  is not an isomorphism. More precisely

$$(2) \quad \mathcal{O}_{Z_s^o(k, N)}(D_{s,s-1}^o(k, N)) \cong \det(V_1/W_1)^\vee \otimes \det(W_2/V_2).$$

Finally,  $Z_s^o(k, N) \cap Z_{s'}(k, N) = \emptyset$  if  $|s - s'| > 1$ .

*Proof.*  $D_{s,s+1}^o(k, N) \subset Z_s^o(k, N)$  is the locus where  $\dim \ker X = N - k + s + 1$ . This is precisely the locus where  $X : \mathbb{C}^N/W_2 \rightarrow W_1$  does not induce an isomorphism. Forgetting  $X$  gives us a map from  $D_{s,s+1}^o(k, N)$  to the Schubert locus  $\{(V_1, V_2) : \dim V_1 \cap V_2 = k - s + 1\}$ . Since  $GL(N)$  acts transitively on the Schubert locus this map is a fibration. One can check the fibres are irreducible and hence  $D_{s,s+1}^o(k, N)$  is irreducible. Thus we conclude that

$$\mathcal{O}_{Z_s^o(k, N)}(m \cdot D_{s,s+1}^o(k, N)) \cong \det(\mathbb{C}^N/W_2)^\vee \otimes \det(W_1)$$

for some  $m \in \mathbb{N}$  (i.e. we could have  $m > 1$  if scheme-theoretically the locus where  $X : \mathbb{C}^N/W_2 \rightarrow W_1$  fails to be an isomorphism is non-reduced). To see that  $m = 1$  we simply need to check that  $\det(\mathbb{C}^N/W_2)^\vee \otimes \det(W_1)$  is not a multiple of a line bundle on  $Z_s^o(k, N)$ . This can be seen quite easily on the compactification  $Z_s''(k, N)$  (defined in the proof of Lemma 3.1) by showing that its Chern class is primitive. This proves (1). The proof of (2) is similar.

Finally, suppose  $s' = s + l$  where  $l \geq 2$ . The locus  $Z_s(k, N) \cap Z_{s'}(k, N)$  corresponds to  $\dim \ker X \geq N - k + s + l$  and  $\dim V_1 \cap V_2 \geq k - s$ . This means  $\dim \ker X + \dim V_1 \cap V_2 \geq N + l > N + 1$  so the intersection lies outside the open subscheme  $Z_s^o(k, N)$ . This shows  $Z_s^o(k, N) \cap Z_{s'}(k, N) = \emptyset$ . The case  $l \leq -2$  is dealt with similarly.  $\square$

**Corollary 3.4.** *On  $Z_s^o(k, N)$  we have*

$$\det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2) \cong \mathcal{O}_{Z_s^o(k, N)}(D_{s,s+1}^o(k, N) - D_{s,s-1}^o(k, N)).$$

*Proof.* This is a direct consequence of Lemma 3.3 since

$$\begin{aligned} \det(\mathbb{C}^N/W_2)^\vee \det(W_1) \det(V_1/W_1) \det(W_2/V_2)^\vee &\cong \det(\mathbb{C}^N/W_2)^\vee \det(V_1) \\ &\cong (\det(\mathbb{C}^N)^\vee \det(V_2)) \otimes (\det(\mathbb{C}^N) \det(\mathbb{C}^N/V_1)^\vee) \\ &\cong \det(\mathbb{C}^N/V_1)^\vee \det(V_2). \end{aligned}$$

$\square$

**3.2. The kernel  $\mathcal{T}(k, N)$ .** In [CKL3] we defined a kernel  $\mathcal{T}(k, N) \in D(Y(k, N) \times Y(N - k, N))$  as the convolution of a complex

$$\Theta_k(k, N) \rightarrow \Theta_{k-1}(k, N) \rightarrow \cdots \rightarrow \Theta_1(k, N) \rightarrow \Theta_0(k, N).$$

Here  $\Theta_s(k, N) = \mathcal{F}^{(N-2k+s)}(N - k, N) * \mathcal{E}^{(s)}(k, N)[-s]$  where

$$\begin{aligned} \mathcal{E}^{(s)}(k, N) &= \mathcal{O}_{W^s(k-s, N)} \otimes \det(\mathbb{C}^N/V')^{-s} \det(V_1)^s \in D(Y(k, N) \times Y(k - s, N)) \\ \mathcal{F}^{(N-2k+s)}(N - k, N) &= \mathcal{O}_{W^{N-2k+s}(k-s, N)} \otimes \det(V_2/V')^s \in D(Y(k - s, N) \times Y(N - k, N)). \end{aligned}$$

The varieties  $W^s(k - s, N)$  and  $W^{N-2k+s}(k - s, N)$  were defined at the end of section 2.2.

One of the main results of [CKL1], [CKL2] and [CKL3] is that:

**Theorem 3.5.**  $\mathcal{T}(k, N) \in D(Y(k, N) \times Y(N - k, N))$  induces an equivalence of stratified Mukai flops

$$\mathcal{T}(k, N) : D(Y(k, N)) \xrightarrow{\sim} D(Y(N - k, N)).$$

Next we identify  $\mathcal{T}(k, N)$  as an  $S_2$  extension of sheaves.

**Proposition 3.6.** *We have*

$$\Theta_s(k, N) \cong i_* j_* \mathcal{O}_{Z_s^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s[-s] \in D(Y(k, N) \times Y(N - k, N)).$$

*Proof.* By definition we have

$$\Theta_s(k, N) \cong \pi_{13*}(\pi_{12}^* \mathcal{O}_{W^s(k-s, N)} \otimes \pi_{23}^* \mathcal{O}_{W^{N-2k+s}(k-s, N)} \otimes \det(\mathbb{C}^N/V')^{-s} \det(V_1)^s \det(V_2/V')^s).$$

Now

$$\pi_{12}^{-1}(W^s(k - s, N)) \cap \pi_{23}^{-1}(W^{N-2k+s}(k - s, N)) \subset Y(k, N) \times Y(k - s, N) \times Y(N - k, N)$$

is the irreducible scheme

$$\{0 \xrightarrow{k-s} V' \xrightarrow[N-2k+s]{s} \begin{matrix} V_1 & \xrightarrow{N-k} \\ V_2 & k \end{matrix} \mathbb{C}^N : X\mathbb{C}^N \subset V', XV_1 \subset 0, XV_2 \subset 0\}$$

which, as we saw in Lemma 3.1, is the partial resolution  $Z'_s(k, N)$  of  $Z_s(k, N)$ . In particular,  $\dim Z'_s(k, N) = 2k(N - k)$  which is the expected dimension of  $\pi_{12}^{-1}(W^s(k - s, N)) \cap \pi_{23}^{-1}(W^{N-2k+s}(k - s, N))$ . So we have the intersection of two smooth varieties which intersect in the expected dimension. This means  $\pi_{12}^* \mathcal{O}_{W^s(k-s, N)} \otimes \pi_{23}^* \mathcal{O}_{W^{N-2k+s}(k-s, N)} \cong \mathcal{O}_{Z'_s(k, N)}$  (one can check that the intersection is reduced by a local or cohomological calculation).

Now  $\pi_{13}$  forgets  $V'$  and hence maps  $\pi_{13} : Z'_s(k, N) \rightarrow Z_s(k, N)$ . By Corollary 3.2  $\pi_{13*} \mathcal{O}_{Z'_s(k, N)} \cong i_* j_* \mathcal{O}_{Z_s^o(k, N)}$ . Thus, by the projection formula,

$$\pi_{13*}(\mathcal{O}_{Z'_s(k, N)} \otimes \det(\mathbb{C}^N/V')^{-s} \det(V_1)^s \det(V_2/V')^s) \cong i_* j_* \mathcal{O}_{Z_s^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s$$

since

$$\begin{aligned} \det(\mathbb{C}^N/V')^{-s} \det(V_1)^s \det(V_2/V')^s &\cong \det(\mathbb{C}^N)^{-s} \det(V')^s \det(V_1)^s \det(V_2)^s \det(V')^{-s} \\ &\cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s. \end{aligned}$$

□

**Theorem 3.7.** *There exists  $\mathbb{C}^*$ -equivariant line bundle  $\mathcal{L}(k, N)$  on  $Z^o(k, N)$  such that  $\mathcal{T}(k, N) \cong i_* j_* \mathcal{L}(k, N)$  where  $Z^o(k, N) \xrightarrow{j} Z(k, N) \xrightarrow{i} Y(k, N) \times Y(N - k, N)$ . This line bundle is uniquely determined by its restrictions*

$$\mathcal{L}(k, N)|_{Z_s^o(k, N)} \cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s \otimes \mathcal{O}_{Z_s^o(k, N)}(D_{s, s+1}^o(k, N))$$

and the fact it is  $\mathbb{C}^*$ -equivariant.

*Proof.* The convolution of  $\Theta_*(k, N)$  is obtained by taking repeated cones starting on the left. More precisely, if we consider the convolution of the partial complex

$$\mathcal{T}_{\geq s}(k, N) := \text{Conv}(\Theta_k(k, N) \rightarrow \cdots \rightarrow \Theta_s(k, N))$$

then we have the exact triangle

$$\mathcal{T}_{\geq s}(k, N) \rightarrow \Theta_{s-1}(k, N) \rightarrow \mathcal{T}_{\geq s-1}(k, N).$$

It is easy to see by induction that  $\mathcal{T}_{\geq s}(k, N)$  is a sheaf shifted into cohomological degree  $s$ . Moreover, each  $\Theta_s(k, N)$  is an  $S_2$  sheaf with reduced scheme theoretic support. So by Proposition 3.6 and Lemma 2.4 (and induction) each  $\mathcal{T}_{\geq s}(k, N)$  is an  $S_2$  sheaf with reduced scheme theoretic support. In particular, so is  $\mathcal{T}(k, N)$ .

Since each  $\mathcal{T}_{\geq s}(k, N)$  has reduced scheme theoretic support it is the pushforward of a sheaf on  $Z(k, N)$ . For the rest of the proof we will work on  $Z(k, N)$  and to simplify notation we will think of  $\mathcal{T}(k, N)$ ,  $\mathcal{T}_{\geq s}(k, N)$  or  $\Theta_s(k, N)$  as lying on  $Z(k, N)$ .

Now by Lemma 3.1 the complement  $Z(k, N) \setminus Z^o(k, N)$  has codimension at least four. So by Proposition 2.2, to show  $\mathcal{T}(k, N) \cong j_*\mathcal{L}(k, N)$  it suffices to show  $j^*\mathcal{T}(k, N) \cong \mathcal{L}(k, N)$ .

We will show by decreasing induction that there exists line bundle  $\mathcal{L}_{\geq s}(k, N)$  on  $Z_{\geq s}^o(k, N)$  such that

$$j^*\mathcal{T}_{\geq s}(k, N) \cong \mathcal{L}_{\geq s}(k, N)[-s]$$

where  $Z_{\geq s}^o(k, N) := Z_k^o(k, N) \cup \cdots \cup Z_s^o(k, N)$ . Moreover,  $\mathcal{L}_{\geq s}(k, N)$  is uniquely determined by its restrictions

$$\mathcal{L}_{\geq s}(k, N)|_{Z_{s'}^o(k, N)} \cong \det(\mathbb{C}^N/V_1)^{-s'} \det(V_2)^{s'} \otimes \mathcal{O}_{Z_{s'}^o(k, N)}(D_{s', s'+1}^o(k, N))$$

if  $k \geq s' > s$  and

$$\mathcal{L}_{\geq s}(k, N)|_{Z_s^o(k, N)} \cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s.$$

The base case  $s = k$  follows by Proposition 3.6. To prove the induction step we look at the exact triangle

$$(3) \quad j^*\mathcal{T}_{\geq s}(k, N) \xrightarrow{d_s} j^*\Theta_{s-1}(k, N) \rightarrow j^*\mathcal{T}_{\geq s-1}(k, N)$$

where  $j^*\Theta_{s-1}(k, N) \cong \mathcal{O}_{Z_{s-1}^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s+1} \det(V_2)^{s-1}[-s+1]$  by Proposition 3.6. Now

$$Z_{\geq s}^o(k, N) \cap Z_{s-1}(k, N) = Z_s^o(k, N) \cap Z_{s-1}(k, N) = D_{s-1, s}^o(k, N)$$

in the notation of Lemma 3.3. Thus we get

$$\begin{aligned} & \text{Hom}(j^*\mathcal{T}_{\geq s}(k, N), j^*\Theta_{s-1}(k, N)) \\ & \cong \text{Hom}(\mathcal{L}_{\geq s}(k, N)[-s], \mathcal{O}_{Z_{s-1}^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s+1} \det(V_2)^{s-1}[-s+1]) \\ & \cong \text{Hom}(\mathcal{L}_{\geq s}(k, N)|_{Z_s^o(k, N)}, \mathcal{O}_{Z_{s-1}^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s+1} \det(V_2)^{s-1}[1]) \\ & \cong \text{Hom}(\mathcal{O}_{Z_s^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s, \mathcal{O}_{Z_{s-1}^o(k, N)} \otimes \det(\mathbb{C}^N/V_1)^{-s+1} \det(V_2)^{s-1}[1]) \\ & \cong \text{Hom}(\mathcal{O}_{Z_s^o(k, N)}, \mathcal{O}_{Z_{s-1}^o(k, N)} \otimes \det(\mathbb{C}^N/V_1) \det(V_2)^\vee[1]) \\ & \cong \text{Hom}(\mathcal{O}_{Z_s^o(k, N)}, \mathcal{O}_{Z_{s-1}^o(k, N)}(-D_{s-1, s}^o(k, N) + D_{s-1, s-2}^o(k, N))[1]) \\ & \cong \text{Hom}(\mathcal{O}_{D_{s-1, s}^o(k, N)}, \mathcal{O}_{D_{s-1, s}^o(k, N)}) \cong H^0(\mathcal{O}_{D_{s-1, s}^o(k, N)}) \end{aligned}$$

where the second isomorphism follows since  $Z_{s'}^o(k, N) \cap Z_{s-1}^o(k, N) = \emptyset$  if  $s' > s$ , the second last isomorphism follows by Lemma 3.3 and the last isomorphism follows by Lemma 3.8.

Since the complement of  $D_{s-1, s}^o(k, N)$  in

$$\begin{aligned} D_{s-1, s}(k, N) &= Z_{s-1}(k, N) \cap Z_s(k, N) \\ &= \{(X, V_1, V_2) \in Z(k, N) : \dim \ker X \geq N - k + s \text{ and } \dim V_1 \cap V_2 \geq k - s + 1\} \end{aligned}$$

has codimension at least two it follows  $H^0(\mathcal{O}_{D_{s-1,s}^o(k,N)}) \cong H^0(D_{s-1,s}(k,N))$ . We would like to show there are only the constant functions on  $D_{s-1,s}(k,N)$ . This is not true but there is one trick we can apply. All of our constructions so far have been  $\mathbb{C}^*$ -equivariant with respect to the  $\mathbb{C}^*$ -actions defined in section 2.4. So the map

$$d_s \in \text{Hom}(j^* \mathcal{T}_{\geq s}(k,N), j^* \Theta_{s-1}(k,N)) \cong H^0(D_{s-1,s}(k,N))$$

must be  $\mathbb{C}^*$ -equivariant. But  $\mathbb{C}^*$  acts by scaling  $X \in \text{End}(\mathbb{C}^N)$  so the closure of every  $\mathbb{C}^*$  orbit intersects the locus where  $X = 0$ . Thus

$$H_{\mathbb{C}^*}^0(D_{s-1,s}(k,N)) \cong H^0(\{(V_1, V_2) \in \mathbb{G}(k,N) \times \mathbb{G}(N-k,N) : \dim V_1 \cap V_2 \geq k-s+1\}) \cong \mathbb{C}.$$

We conclude that any  $\mathbb{C}^*$ -equivariant map such as  $d_s$  is unique (up to a multiple). Note also that  $d_s \neq 0$  because if  $d_s = 0$  then  $\mathcal{T}(k,N)$  is decomposable and would not induce an equivalence.

On the other hand, notice that

$$\begin{aligned} & j^* \Theta_{s-1}(k,N) \otimes \mathcal{O}_{Z_{s-1}^o(k,N)}(D_{s-1,s}^o(k,N))[s-1] \\ & \cong \det(\mathbb{C}^N/V_1)^{-s+1} \det(V_2)^{s-1} \otimes \mathcal{O}_{Z_{s-1}^o(k,N)}(D_{s-1,s}^o(k,N)) \\ & \cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s \otimes \left( \det(\mathbb{C}^N/V_1) \det(V_2)^\vee \otimes \mathcal{O}_{Z_{s-1}^o(k,N)}(D_{s-1,s}^o(k,N)) \right) \\ & \cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s \otimes \mathcal{O}_{Z_{s-1}^o(k,N)}(D_{s-1,s-2}^o(k,N)) \end{aligned}$$

where the last isomorphism follows from Corollary 3.4. Since  $D_{s-1,s-2}^o(k,N) \cap D_{s-1,s}^o(k,N) = \emptyset$  the restriction of this line bundle to  $D_{s-1,s}^o(k,N)$  is the line bundle  $\det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s$ . This is the same as the restriction of  $\mathcal{L}_{\geq s}(k,N)$  to  $D_{s-1,s}^o(k,N) \subset Z_s^o(k,N)$ . Since, as we saw above,  $H_{\mathbb{C}^*}^0(D_{s-1,s}(k,N)) \cong \mathbb{C}$  there is a unique  $\mathbb{C}^*$ -equivariant line bundle, namely  $\mathcal{L}_{\geq s-1}(k,N)$ , which restricts as above.

Finally, consider the standard short exact sequence

$$\mathcal{O}_{Z_{s-1}^o(k,N)}(-D_{s-1,s}^o(k,N)) \rightarrow \mathcal{O}_{Z_{\geq s-1}^o(k,N)} \rightarrow \mathcal{O}_{Z_s^o(k,N)}$$

where the right hand map is restriction. In an analogous manner we get the sequence

$$j^* \Theta_{s-1}(k,N)[s-1] \rightarrow \mathcal{L}_{\geq s-1}(k,N) \rightarrow \mathcal{L}_{\geq s}(k,N)$$

which, after shifting by  $[-s+1]$  and moving the right term over to the left side, gives

$$j^* \mathcal{T}_{\geq s}(k,N) \xrightarrow{d} j^* \Theta_{s-1}(k,N) \rightarrow \mathcal{L}_{\geq s-1}[-s+1].$$

Notice that the map  $d$  is non-zero and that everything in this final calculation is again  $\mathbb{C}^*$ -equivariant. So  $d = d_s$  up to some non-zero multiple and comparing with (3) we get  $j^* \mathcal{T}_{\geq s-1}(k,N) \cong \mathcal{L}_{\geq s-1}[-s+1]$  (which completes the induction).  $\square$

**Lemma 3.8.** *Consider two smooth schemes  $Z_1$  and  $Z_2$  intersecting along a divisor  $D = Z_1 \cap Z_2$ . Then  $\text{Hom}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}(-D)[1]) \cong \text{Hom}(\mathcal{O}_D, \mathcal{O}_D)$ .*

*Proof.* See, for example, Corollary 4.8 of [CK1].  $\square$

#### 4. EQUIVALENCES OF STRATIFIED ATIYAH FLOPS

If we restrict to  $\tilde{Z}^o(k,N)$ , which is smooth, each  $Z_s^o(k,N) \subset \tilde{Z}^o(k,N)$  defines a line bundle. Notice that  $Z_s(k,N) \subset \tilde{Z}(k,N)$  is in general only a divisor (it may fail to be Cartier). The following is the main result of this paper. Recall the natural inclusions

$$\tilde{Z}^o(k,N) \xrightarrow{\tilde{j}} \tilde{Z}(k,N) \xrightarrow{\tilde{i}} \tilde{Y}(k,N) \times_{\mathbb{A}^1} \tilde{Y}(N-k,N).$$

**Theorem 4.1.** Denote by  $\tilde{\mathcal{L}}(k, N)$  the line bundle  $\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k \binom{s+1}{2} Z_s^o(k, N))$ . Then

$$\tilde{T}(k, N) := \tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}(k, N) \in D(\tilde{Y}(k, N) \times_{\mathbb{A}^1} \tilde{Y}(N-k, N))$$

induces an equivalence  $\tilde{T}(k, N) : D(\tilde{Y}(k, N)) \xrightarrow{\sim} D(\tilde{Y}(N-k, N))$  of stratified Atiyah flops. Moreover, the restriction of  $\tilde{T}(k, N)$  to  $Y(k, N) \times Y(N-k, N)$  is  $\mathcal{T}(k, N)$ .

*Proof.* Over any non-zero  $x \in \mathbb{A}^1$  the fibre  $\tilde{Z}^o(k, N)_x \cong \tilde{Z}(k, N)_x$  maps isomorphically down to  $\tilde{Y}(k, N)_x$  and  $\tilde{Y}(N-k, N)_x$  and so  $\tilde{T}(k, N)_x$  induces an equivalence. So to show that  $\tilde{T}(k, N)$  is an equivalence it suffices, by Lemma 4.2 below, to show that  $\tilde{T}(k, N)$  restricts to  $\mathcal{T}(k, N)$  over the central fibre  $Y(k, N) \times Y(N-k, N)$ .

Consider the following commutative diagram

$$\begin{array}{ccccc} Z^o(k, N) & \xrightarrow{j} & Z(k, N) & \xrightarrow{i} & Y(k, N) \times Y(N-k, N) \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ \tilde{Z}^o(k, N) & \xrightarrow{\tilde{j}} & \tilde{Z}(k, N) & \xrightarrow{i} & \tilde{Y}(k, N) \times_{\mathbb{A}^1} \tilde{Y}(N-k, N) \end{array}$$

where the  $f$ 's denote the natural inclusions of the central fibre. We have

$$\begin{aligned} f_3^* \tilde{T}(k, N) &\cong f_3^* \tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}(k, N) \cong i_* f_2^* \tilde{j}_* \tilde{\mathcal{L}}(k, N) \\ &\cong i_* j_* j^* f_2^* \tilde{j}_* \tilde{\mathcal{L}}(k, N) \cong i_* j_* f_3^* j^* \tilde{j}_* \tilde{\mathcal{L}}(k, N) \cong i_* j_* f_3^* \mathcal{L}(k, N) \end{aligned}$$

where the second isomorphism on the first row follows since  $\tilde{Z}(k, N)$  is flat over  $\mathbb{A}^1$  and the first isomorphism on the second row is because  $\mathcal{L}(k, N)$  and hence  $f_2^* \tilde{j}_* \tilde{\mathcal{L}}(k, N)$  is an  $S_2$  sheaf (we also used Lemma 2.2 which applies to both  $j$  and  $\tilde{j}$ ). Since  $\mathcal{T}(k, N) \cong i_* j_* \mathcal{L}(k, N)$  it follows that what we really need to check is that  $f_3^* \tilde{\mathcal{L}}(k, N) \cong \mathcal{L}(k, N)$ .

Notice that everything is still  $\mathbb{C}^*$ -equivariant so  $\mathcal{L}(k, N)$  is uniquely determined by its restrictions to  $Z_s^o(k, N)$ . Hence it suffices to show that

$$\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k \binom{s+1}{2} Z_s^o(k, N))|_{Z_s^o(k, N)} \cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s \otimes \mathcal{O}_{Z_s^o(k, N)}(D_{s, s+1}^o(k, N)).$$

To see this we use two facts. The first is that  $Z_{s'}^o(k, N) \cap Z_s^o(k, N) = \emptyset$  if  $|s - s'| > 1$  so that  $\mathcal{O}_{\tilde{Z}^o(k, N)}(Z_{s'}^o(k, N))|_{Z_s^o(k, N)}$  is trivial if  $|s - s'| > 1$ . The second fact is that  $\sum_{s=0}^k Z_s^o(k, N)$  is equal to the class of a fibre meaning that the line bundle  $\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k Z_s^o(k, N))$  is trivial. Putting them together we get

$$\begin{aligned} &\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k \binom{s+1}{2} Z_s^o(k, N))|_{Z_s^o(k, N)} \\ &\cong \mathcal{O}_{\tilde{Z}^o(k, N)}(\binom{s}{2} Z_{s-1}^o(k, N) + \binom{s+1}{2} Z_s^o(k, N) + \binom{s+2}{2} Z_{s+1}^o(k, N))|_{Z_s^o(k, N)} \\ &\cong \mathcal{O}_{\tilde{Z}^o(k, N)}(-s Z_{s-1}^o(k, N) + (s+1) Z_{s+1}^o(k, N))|_{Z_s^o(k, N)} \\ &\cong \mathcal{O}_{Z_s^o(k, N)}(-s D_{s, s-1}^o(k, N) + (s+1) D_{s, s+1}^o(k, N)) \\ &\cong \det(\mathbb{C}^N/V_1)^{-s} \det(V_2)^s \otimes \mathcal{O}_{Z_s^o(k, N)}(D_{s, s+1}^o(k, N)) \end{aligned}$$

where the first isomorphism uses the first fact, the second uses the second fact and the last uses Corollary 3.4.  $\square$

**Lemma 4.2.** *Let  $\tilde{Y} \rightarrow B$  and  $\tilde{Y}' \rightarrow B$  be two families of smooth varieties over some base  $B$  and let  $\mathcal{T} \in \tilde{Y} \times_B \tilde{Y}'$  be a relative kernel. If  $\mathcal{T}|_{\tilde{Y}_t \times \tilde{Y}'_t}$  induces an equivalence  $D(\tilde{Y}_t) \xrightarrow{\sim} D(\tilde{Y}'_t)$  for every  $t \in B$  then  $\mathcal{T}$  induces an equivalence  $D(\tilde{Y}) \xrightarrow{\sim} D(\tilde{Y}')$ .*

*Proof.* This follows by Proposition 3.2 of [Sz].  $\square$

## 5. GENERAL REMARKS

**5.1. Choice of line bundles.** The choice of line bundles  $\mathcal{L}(k, N)$  and  $\tilde{\mathcal{L}}(k, N)$  in Theorems 3.7 and 4.1 might seem a little random. One might ask, for instance, what (if anything) goes wrong if we take  $\tilde{\mathcal{L}}(k, N)$  to be the trivial line bundle.

Namikawa showed in [N2] that in the case of  $\tilde{Y}(2, 4)$  the trivial line bundle on  $\tilde{Z}(2, 4)$  (or  $Z(2, 4)$ ) fails to induce an equivalence. From our point of view, the reason for this should be attributed to the fact that  $i_* j_* \tilde{\mathcal{L}}(k, N)$  can be shown to be Cohen-Macaulay while  $i_* j_* \mathcal{O}_{\tilde{Z}^o(k, N)}$  fails to be Cohen-Macaulay (if  $k \neq 1, N - 1$ ).

On the other hand, we can tensor  $\tilde{\mathcal{L}}(k, N)$  by the restriction  $L$  of any line bundle from  $\tilde{Z}(k, N)$  and then  $i_* j_* (\tilde{\mathcal{L}}(k, N) \otimes L)$  will still induce an equivalence. An interesting choice is  $L := \det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2)$  because:

**Lemma 5.1.** *We have  $L \cong \mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=1}^k s Z_s^o(k, N))$ .*

*Proof.* The natural inclusion map  $V_2 \rightarrow \mathbb{C}^N/V_1$  on  $\tilde{Z}^o(k, N)$  is an isomorphism over any general fibre  $x \neq 0$  because  $V_1 \cap V_2 = 0$ . So  $\det(V_2) \cong \det(\mathbb{C}^N/V_1)$  which means that for  $x \neq 0$

$$L_x \cong \mathcal{O}_{\tilde{Z}^o(k, N)_x} \cong \mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=1}^k s Z_s^o(k, N))_x.$$

Since every line bundle on the base  $\mathbb{A}^1$  is trivial it remains to show that  $L'$  and  $\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=1}^k s Z_s^o(k, N))$  restrict to isomorphic line bundles over  $Z^o(k, N)$ . Since both line bundles are  $\mathbb{C}^*$ -equivariant, the same argument as in the proof of Theorem 3.7 implies that it suffices to show

$$L|_{Z_s^o(k, N)} \cong \mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=1}^k s Z_s^o(k, N))|_{Z_s^o(k, N)}$$

for every  $s$ . Now we have

$$\begin{aligned} & \mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=1}^k s Z_s^o(k, N))|_{Z_s^o(k, N)} \\ & \cong \mathcal{O}_{\tilde{Z}^o(k, N)}((s-1)Z_{s-1}^o(k, N) + sZ_s^o(k, N) + (s+1)Z_{s+1}^o(k, N))|_{Z_s^o(k, N)} \\ & \cong \mathcal{O}_{\tilde{Z}^o(k, N)}(-Z_{s-1}^o(k, N) + Z_{s+1}^o(k, N))|_{Z_s^o(k, N)} \\ & \cong \mathcal{O}_{Z_s^o(k, N)}(-D_{s, s-1}^o(k, N) + D_{s, s+1}^o(k, N)) \\ & \cong L|_{Z_s^o(k, N)} \end{aligned}$$

where the first isomorphism uses that  $Z_s^o(k, N) \cap Z_{s'}^o(k, N) = \emptyset$  if  $|s - s'| > 1$ , the second uses that  $\mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k s Z_s^o(k, N))$  is trivial while the last uses Corollary 3.4.  $\square$

Consequently, since  $\binom{s+1}{2} - s = \binom{s}{2}$ , we find that:

**Corollary 5.2.** *The kernel  $\mathcal{S}(k, N) := i_* j_* (\tilde{\mathcal{L}}(k, N) \otimes L^\vee) \cong i_* j_* \mathcal{O}_{\tilde{Z}^o(k, N)}(\sum_{s=0}^k \binom{s}{2} Z_s^o(k, N))$  induces an equivalence of stratified Atiyah flops.*

Notice that when  $k = 1$  then  $\tilde{\mathcal{L}}(1, N) \otimes L^\vee \cong \mathcal{O}_{\tilde{Z}^o(1, N)}$  which explains why the structure sheaf of the natural correspondence for the standard Atiyah flop *does* induce an equivalence (as proven in [Ka1] or [N1]).

**5.2. Inverses.** It is instructive to identify the inverse of  $\mathbb{T}(k, N)$ . The inverse is induced by the left (or right) adjoint of  $\mathcal{T}(k, N)$ . One way to do this is to write  $\mathcal{T}(k, N)$  as the convolution of  $\Theta_*(k, N)$  and take left adjoints to get

$$\mathcal{T}(k, N)_L \cong \text{Conv}(\Theta_0(k, N)_L \rightarrow \Theta_1(k, N)_1 \rightarrow \cdots \rightarrow \Theta_k(k, N)_L)$$

(where one takes the right convolution). Then one can identify this object as the pushforward via  $(i \circ j)$  of some line bundle as we did in Theorem 3.7.

On the other hand, we can find  $\mathcal{T}(k, N)_L$  more simply as follows. We know that

$$\begin{aligned} \Theta_s(k, N)_L &= \mathcal{E}^{(s)}(k, N)_L * \mathcal{F}^{(N-2k+s)}(N-k, N)_L[s] \\ &\cong \mathcal{F}^{(s)}(k, N)[-s(N-2k+s)] * \mathcal{E}^{(N-2k+s)}(N-k, N)[(N-2k+s)(s)][s] \\ &\cong \mathcal{F}^{(s)}(k, N) * \mathcal{E}^{(N-2k+s)}(N-k, N)[s] \end{aligned}$$

where we use that the left and right adjoints of any  $\mathcal{E}^{(s)}$  is  $\mathcal{F}^{(s)}$  up to some shift (for the precise shift in general see [CKL3]). The same argument used in the proof Proposition 3.6 shows that  $\Theta_s(k, N)_L$  is a sheaf in cohomological degree  $(-s)$  and then the right convolution of  $\Theta_*(k, N)_L$  is a sheaf.

This sheaf is  $S_2$  so it is the pushforward of its restriction to  $Z^o(k, N)$ . Now  $Z^o(k, N)$  is Cohen-Macaulay so this sheaf must be

$$\mathcal{O}_{Z^o(k, N)}^\vee \otimes \mathcal{L}(k, N)^\vee \otimes \pi_1^* \omega_{Y(k, N)}[\dim Y(k, N)] \cong \omega_{Z^o(k, N)} \otimes \mathcal{L}(k, N)^\vee$$

where we use that the canonical bundle of  $Y(k, N) = T^*\mathbb{G}(k, N)$  is trivial.

**Theorem 5.3.** *The inverse of  $\tilde{\mathbb{T}}(k, N)$  is induced by*

$$\tilde{\mathcal{T}}(k, N)_L \cong \tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}'(k, N) \in D(\tilde{Y}(k, N) \times_{\mathbb{A}^1} \tilde{Y}(N-k, N))$$

where

$$\tilde{\mathcal{L}}'(k, N) := \mathcal{O}_{\tilde{Z}^o(k, N)} \left( \sum_{s=0}^k \binom{s}{2} Z_s^o(k, N) \right) \otimes (\det(\mathbb{C}^N/V_1)^\vee \det(V_2))^{\otimes (N-2k)}.$$

*The inverse of  $\mathbb{T}(k, N)$  is induced by the restriction of  $\tilde{\mathcal{T}}(k, N)_L$  to  $Y(k, N) \times Y(N-k, N)$ .*

*Proof.* The restriction of  $\tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}'(k, N)$  to a non-central fibre is the structure sheaf  $\mathcal{O}_{\tilde{Z}^o(k, N)_x}$  since, as in the proof of Lemma 5.1,  $(X - x \cdot \text{id}) : V_2 \xrightarrow{\sim} \mathbb{C}^N/V_1$ . Notice that this gives the inverse to the kernel induced by  $\mathcal{O}_{\tilde{Z}^o(k, N)_x}$ . This is because, when  $x \neq 0$ ,  $\pi_{12}^{-1}(\tilde{Z}^o(k, N)_x)$  and  $\pi_{23}^{-1}(\tilde{Z}^o(k, N)_x)$  intersect transversely and the intersection maps one-to-one via  $\pi_{13}$  to the diagonal in  $Y(k, N)_x \times Y(N-k, N)_x$ . Hence, if  $x \neq 0$ , the inverse of  $\tilde{\mathbb{T}}(k, N)|_x$  is  $(\tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}'(k, N))|_x$ . It remains to show this when  $x = 0$ .

To do this we need to show that

$$(4) \quad \tilde{i}_* \tilde{j}_* \tilde{\mathcal{L}}'(k, N)|_{Y(k, N) \times Y(N-k, N)} \cong i_* j_* (\omega_{Z^o(k, N)} \otimes \mathcal{L}(k, N)^\vee).$$

But

$$\omega_{Z^o(k, N)} \otimes \mathcal{L}(k, N) \cong (\omega_{\tilde{Z}^o(k, N)} \otimes \tilde{\mathcal{L}}(k, N)^\vee)|_{Y(k, N) \times Y(N-k, N)}$$



and by Proposition 5.5 we have

$$\begin{aligned}
& \omega_{\tilde{Z}^o(k,N)} \otimes \tilde{\mathcal{L}}(k,N)^\vee \\
& \cong \mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k s^2 Z_s^o(k,N) \right) \otimes L^{\otimes(N-2k)} \otimes \mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k -\binom{s+1}{2} Z_s^o(k,N) \right) \\
& \cong \mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k \binom{s}{2} Z_s^o(k,N) \right) \otimes L^{\otimes(N-2k)}
\end{aligned}$$

where, as before,  $L = \det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2)$ . This proves (4).  $\square$

**Remark 5.4.** Notice that the difference between  $\mathcal{L}(k,N)$  and its adjoint  $\mathcal{L}'(k,N)$  is

$$(\det(\mathbb{C}^N/V_1) \otimes \det(V_2))^{\otimes(N-2k-1)}$$

which is a global line bundle. Even more, this line bundle is the tensor product of pullbacks of line bundles from  $\tilde{Y}(k,N)$  and  $\tilde{Y}(N-k,N)$ . Thus  $\mathcal{T}(k,N)$  and its inverse differ by conjugation. This fact cannot be seen directly from the definition of  $\mathcal{L}(k,N)$  coming from categorical  $\mathfrak{sl}_2$  actions. Although this seems like a strange observation, it should help in constructing *affine* braid group actions, extending the work in [CK4].

**Proposition 5.5.** *The dualizing sheaf of  $\tilde{Z}^o(k,N)$  is*

$$\omega_{\tilde{Z}^o(k,N)} \cong \mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k s^2 Z_s^o(k,N) \right) \otimes (\det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2))^{\otimes(N-2k)}.$$

*Proof.* The right hand side restricts to the trivial line bundle on any non-central fibre  $\tilde{Z}^o(k,N)$ . This agrees with  $\omega_{\tilde{Z}^o(k,N)_x} \cong \omega_{\tilde{Y}(k,N)_x}$  which is trivial since  $\tilde{Y}(k,N)_x$  is actually obtained from  $T^*\mathbb{G}(k,N)$  by a hyper-Kähler rotation. Since  $\text{Pic}(\mathbb{A}^1)$  is trivial we are reduced to showing that

$$\omega_{Z^o(k,N)} \cong (\mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k s^2 Z_s^o(k,N) \right) \otimes L^{\otimes(N-2k)})|_{Z^o(k,N)}$$

where  $L = \det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2)$ .

$Z^o(k,N)$  is made up of  $k+1$  smooth components  $Z_s^o(k,N)$ . Now

$$\omega_{Z^o(k,N)}|_{Z_s^o(k,N)} \cong \omega_{Z_s^o(k,N)}(D_{s,s-1}^o(k,N) + D_{s,s+1}^o(k,N))$$

so it suffices to show that

$$(5) \quad \omega_{Z_s^o(k,N)}(D_{s,s-1}^o(k,N) + D_{s,s+1}^o(k,N)) \cong (\mathcal{O}_{\tilde{Z}^o(k,N)} \left( \sum_{s=0}^k s^2 Z_s^o(k,N) \right) \otimes L^{\otimes(N-2k)})|_{Z_s^o(k,N)}$$

for every  $s = 0, \dots, k$ . But  $\mathcal{O}_{\tilde{Z}^o(k,N)}(Z_s^o(k,N)) \cong \mathcal{O}_{\tilde{Z}^o(k,N)}(\sum_{s' \neq s} -Z_{s'}^o(k,N))$  and  $Z_s^o(k,N) \cap Z_{s'}^o(k,N) = \emptyset$  if  $|s - s'| > 1$ . So the right hand side of (5) equals

$$\begin{aligned}
& \mathcal{O}_{Z_s^o(k,N)}(((s-1)^2 - s^2)D_{s,s-1}^o(k,N) + ((s+1)^2 - s^2)D_{s,s+1}^o(k,N)) \otimes L^{\otimes(N-2k)} \\
& \cong \mathcal{O}_{Z_s^o(k,N)}((-2s+1)D_{s,s-1}^o(k,N) + (2s+1)D_{s,s+1}^o(k,N)) \otimes L^{\otimes(N-2k)} \\
& \cong \mathcal{O}_{Z_s^o(k,N)}(D_{s,s-1}^o(k,N) + D_{s,s+1}^o(k,N)) \otimes L^{\otimes(N-2k+2s)}
\end{aligned}$$

which by Lemma 5.6 equals the left hand side of (5) (the second isomorphism above follows from Corollary 3.4 which states that  $\mathcal{O}_{Z_s^o(k,N)}(-D_{s,s-1}^o(k,N) + D_{s,s+1}^o(k,N)) \cong L$ ).  $\square$

**Lemma 5.6.** *The dualizing sheaf of  $Z_s^o(k, N)$  is*

$$\omega_{Z_s^o(k, N)} \cong (\det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2))^{\otimes(N-2k+2s)}.$$

*Proof.* As we saw in the proof of Lemma 3.1  $Z_s(k, N)$  has a natural resolution

$$Z_s''(k, N) := \{0 \xrightarrow{k-s} W_1 \xrightarrow[N-2k+s]{s} \begin{matrix} V_1 \\ V_2 \end{matrix} \xrightarrow[s]{N-2k+s} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\}$$

where the map  $\pi : Z_s''(k, N) \rightarrow Z_s(k, N)$  is an isomorphism over  $Z_s^o(k, N)$ . Hence  $\omega_{Z_s^o(k, N)} \cong \omega_{Z_s''(k, N)}|_{Z_s^o(k, N)}$ .

Now consider the projection map

$$p : Z_s''(k, N) \rightarrow \{0 \xrightarrow{k-s} W_1 \xrightarrow{N-2k+2s} W_2 \xrightarrow{k-s} \mathbb{C}^N : X\mathbb{C}^N \subset W_1, XW_2 \subset 0\} \cong W^{N-2k+2s}(k-s, N)$$

given by forgetting  $V_1$  and  $V_2$ . This is a  $\mathbb{G}(s, W_2/W_1) \times \mathbb{G}(N-2k+s, W_2/W_1)$  fibration so the relative cotangent bundle is

$$\begin{aligned} \omega_p &\cong (\det(V_1/W_1)^{N-2k+s} \otimes \det(W_2/V_1)^{-s}) \otimes (\det(V_2/W_1)^s \otimes \det(W_2/V_2)^{-N+2k-s}) \\ &\cong (\det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2))^{N-2k+2s} \otimes (\det(\mathbb{C}^N/W_2)^\vee \otimes \det(W_1))^{-N+2k-2s}. \end{aligned}$$

On the other hand,

$$W^{N-2k+2s}(k-s, N) \subset \{0 \xrightarrow{k-s} W_1 \xrightarrow{N-2k+2s} W_2 \xrightarrow{k-s} \mathbb{C}^N : XW_2 \subset 0\}$$

is carved out by the section  $X : \mathbb{C}^N/W_2 \rightarrow W_1$ . Using this fact it is straight-forward to check that

$$\omega_{W^{N-2k+2s}(k-s, N)} \cong (\det(\mathbb{C}^N/W_2)^\vee \otimes \det(W_1))^{N-2k+2s}.$$

Thus we get  $\omega_{Z_s''(k, N)} \cong \omega_p \otimes \omega_{W^{N-2k+2s}(k-s, N)} \cong (\det(\mathbb{C}^N/V_1)^\vee \otimes \det(V_2))^{N-2k+2s}$  and the result follows.  $\square$

**5.3. Kawamata's equivalence for  $T^*\mathbb{G}(2, 4)$ .** In [Ka2] Kawamata studies stratified Atiyah flops for  $\tilde{Y}(2, N) = T^*\mathbb{G}(2, N)$ . Incidentally, Kawamata also calls these stratified Mukai flops (the terminology of stratified Atiyah flops is from [N2]).

Kawamata constructs two functors  $\Psi, \Phi : D(\tilde{Y}(2, N)) \rightarrow D(\tilde{Y}(N-2, N))$  which are adjoints of each other. He shows ([Ka2] Theorem 3.5) that when  $N = 4$  these functors induce equivalences. We would like to briefly outline his construction and explain why:

**Proposition 5.7.** *The kernel  $\mathcal{S}(2, N) = i_* j_* \mathcal{O}_{\tilde{Z}^o(2, N)}(\sum_{s=0}^2 \binom{s}{2} Z_s^o(2, N))$  and its (left or right) adjoint induce Kawamata's functors  $\Psi, \Phi : D(\tilde{Y}(2, N)) \rightarrow D(\tilde{Y}(N-2, N))$ .*

**Remark 5.8.** By Corollary 5.2 the kernel  $\mathcal{S}(2, N)$  induces an equivalence. Hence Kawamata's functors  $\Psi$  and  $\Phi$  are equivalences for all  $N$ .

Inside

$$\tilde{Y}(2, N) = \{(X, V, x) : x \in \mathbb{C}, X \in \text{End}(\mathbb{C}^N), 0 \subset V \subset \mathbb{C}^N, \dim(V) = 2 \text{ and } \mathbb{C}^N \xrightarrow{X-x \cdot \text{id}} V \xrightarrow{X+x \cdot \text{id}} 0\}$$

there are two natural strata: namely the locus where  $X = 0$  (isomorphic to  $\mathbb{G}(2, N)$ ) and the locus where  $\text{rank } X \leq 1$  (the locus  $\text{rank } X \leq 2$  is the whole central fibre  $Y(2, N)$ ). Kawamata first blows up the first locus and then the strict transform of the second locus to obtain

$$\tilde{Y}''(2, N) \xrightarrow{f_1} \tilde{Y}'(2, N) \xrightarrow{f_2} \tilde{Y}(2, N).$$

Inside  $\tilde{Y}''(2, N)$  we denote by  $E_1$  the exceptional divisor of  $f_1$  and by  $E_2$  the strict transform of the exceptional divisor of  $f_2$ . Warning: our labeling of divisors does not match precisely that in [Ka2] for reasons that will soon become clear.

Kawamata then blows up  $\tilde{Y}(N-2, N)$  in the same way to obtain  $\tilde{Y}''(N-2, N)$  and identifies this smooth variety with  $\tilde{Y}''(2, N)$ . To summarize, we arrive at the following commutative diagram

$$(6) \quad \begin{array}{ccccc} & \tilde{Y}''(2, N) \cong \tilde{Y}''(N-2, N) & & & \\ & \swarrow f & \downarrow \pi & \searrow f^+ & \\ \tilde{Y}(2, N) & \xleftarrow{\pi_1} \tilde{Z}(2, N) & \xrightarrow{\pi_2} & \tilde{Y}(N-2, N) & \\ & \searrow & & \swarrow & \\ & \tilde{B}(2, N) & & & \end{array}$$

Notice that  $\tilde{Z}(2, N) = \tilde{Y}(2, N) \times_{\tilde{B}(2, N)} \tilde{Y}(N-2, N)$  so the map  $\pi$  exists by the universal property of fibre products. The functors  $\Psi$  and  $\Phi$  are then defined by

$$\begin{aligned} \Psi(\cdot) &:= f_*(f^{+*}(\cdot) \otimes \mathcal{O}_{\tilde{Y}''(2, N)}(E_2)) \\ \Phi(\cdot) &:= f_*^+(f^*(\cdot) \otimes \mathcal{O}_{\tilde{Y}''(2, N)}((2N-5)E_2 + (N-3)E_1)). \end{aligned}$$

Notice that  $Z(2, N)$  contains three components, namely  $Z_s(2, N)$  where  $s = 0, 1, 2$ . Similarly,  $Y''(2, N)$  (the preimage of  $Y(2, N)$  under  $f$ ) also contains three components, namely  $E_1, E_2$  and the proper transform of  $Y(2, N)$  inside  $\tilde{Y}''(2, N)$  (which we denote  $E_0$ ). The map  $\pi$  then maps  $E_s$  to  $Z_s(2, N)$  for  $s = 0, 1, 2$  (the map is generically one-to-one).

From this we see that the complement of  $\tilde{Y}''(2, N)^\circ := \pi^{-1}(\tilde{Z}^\circ(2, N))$  in  $\tilde{Y}''(2, N)$  has codimension at least two (otherwise it would contain a divisor whose restriction to  $Y''(2, N)$  would be another component). Now denote by  $j' : \tilde{Y}''(2, N)^\circ \rightarrow \tilde{Y}''(2, N)$  the natural open immersion. Thus for any  $a, b \in \mathbb{Z}$

$$j'_*(\mathcal{O}_{\tilde{Y}''(2, N)^\circ}(aE_1 + bE_2)) \cong \mathcal{O}_{\tilde{Y}''(2, N)}(aE_1 + bE_2).$$

Using the commutative diagram

$$\begin{array}{ccc} \tilde{Y}''(2, N)^\circ & \xrightarrow{j'} & \tilde{Y}''(2, N) \\ \downarrow \pi_o & & \downarrow \pi \\ \tilde{Z}^\circ(2, N) & \xrightarrow{j} & \tilde{Z}(2, N) \end{array}$$

we find that

$$\begin{aligned} R^0\pi_*\mathcal{O}_{\tilde{Y}''(2, N)}(E_2) &\cong R^0\pi_* \circ j'_*(\mathcal{O}_{\tilde{Y}''(2, N)^\circ}(E_2)) \cong j_* \circ R^0\pi_{o*}(\mathcal{O}_{\tilde{Y}''(2, N)^\circ}(E_2)) \\ &\cong j_*\mathcal{O}_{\tilde{Z}^\circ(2, N)}(Z_2^\circ(2, N)) \cong \mathcal{S}(2, N) \end{aligned}$$

where the third isomorphism follows by the projection isomorphism since  $E_2 \cong \pi_o^*Z_2^\circ(2, N)$  and  $\pi_o$  is a proper map between smooth varieties. Now

$$\begin{aligned} \Psi(\cdot) &\cong f_*(f^{+*}(\cdot) \otimes \mathcal{O}_{\tilde{Y}''(2, N)}(E_2)) \\ &\cong (\pi_{1*}\pi_*)(\pi^*\pi_2^*(\cdot) \otimes \mathcal{O}_{\tilde{Y}''(2, N)}(E_2)) \\ &\cong \pi_{1*}(\pi_2^*(\cdot) \otimes \pi_*(\mathcal{O}_{\tilde{Y}''(2, N)}(E_2))) \\ &\cong \pi_{1*}(\pi_2^*(\cdot) \otimes \mathcal{S}(2, N)) \end{aligned}$$

which proves Proposition 5.7.

As a consistency check one can similarly show that

$$\begin{aligned}
R^0 \pi_* \mathcal{O}_{\tilde{Y}''(2,N)}((2N-5)E_2 + (N-3)E_1) &\cong j_* \mathcal{O}_{\tilde{Z}^o(2,N)}((2N-5)Z_2^o(2,N) + (N-3)Z_1^o(2,N)) \\
&\cong j_* \mathcal{O}_{\tilde{Z}^o(2,N)} \left( \sum_{s=0}^2 \binom{s}{2} Z_s^o(2,N) \right) \otimes (\det(\mathbb{C}^N/V_1) \det(V_2))^{N-3} \\
&\cong \mathcal{S}(2,N)_L
\end{aligned}$$

where we used that  $\det(\mathbb{C}^N/V_1) \otimes \det(V_2) \cong \mathcal{O}_{\tilde{Z}^o(2,N)}(\sum_{s=0}^2 sZ_s^o(2,N))$ . This means that  $\mathcal{S}(2,N)_L$  induces  $\Phi$ . Note that we already knew this because  $\Psi$  and  $\Phi$  were shown to be adjoints in [Ka2].

**Remark 5.9.** This same argument can be generalized from  $k=2$  to arbitrary  $k$ . More precisely, one can iteratively blow up  $\tilde{Y}(k,N)$  starting with the smallest stratum to obtain

$$\tilde{Y}^{(k)}(k,N) \xrightarrow{f_1} \tilde{Y}^{(k-1)}(k,N) \xrightarrow{f_2} \dots \xrightarrow{f_k} \tilde{Y}(k,N).$$

We denote by  $E_s$  the (proper transform of the) exceptional divisor of  $f_s$ .

Then there is an isomorphism  $\tilde{Y}^{(k)}(k,N) \cong \tilde{Y}^{(k)}(N-k,N)$  and a commutative diagram like in (6). The map  $\pi : \tilde{Y}^{(k)}(k,N) \rightarrow \tilde{Z}(k,N)$  identifies the divisors  $E_s$  with  $Z_s^o(k,N)$  for  $s=0, \dots, k$ . Subsequently:

**Proposition 5.10.**

$$R^0 \pi_* (\mathcal{O}_{\tilde{Y}^{(k)}(k,N)}(\sum_{s=0}^k \binom{s}{2} E_s)) \cong j_* \mathcal{O}_{\tilde{Z}^o(k,N)}(\sum_{s=0}^k \binom{s}{2} Z_s^o(k,N))$$

and thus both kernels induce equivalences  $D(\tilde{Y}(k,N)) \xrightarrow{\sim} D(\tilde{Y}(N-k,N))$ .

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*E-mail address:* [scautis@math.columbia.edu](mailto:scautis@math.columbia.edu)

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY